MATH5011 Exercise 1

(1) Show that every open set in $\mathbb{R}$ can be written as a countable union of mutually disjoint open intervals. Hint: First show that every point x in this open set is contained in a largest open interval I_x. Next, for any x, y, I_x and I_y either coincide and disjoint. Finally, argue there are at most countably many such intervals.

(2) Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\Psi(f, g)$ are measurable for any measurable functions f, g. This result contains Proposition 1.3 as a special case.

(3) Show that $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}([a, b])$ is measurable for all $a, b \in \overline{\mathbb{R}}$.

(4) Let f, g, $f_k, k \geq 1$, be measurable functions from X to $\overline{\mathbb{R}}$.

(a) Show that $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable sets.

(b) Show that $\{x : \lim_{k\to\infty} f_k(x)$ exists and is finite$\}$ is measurable.

(5) There are two conditions (i) and (ii) in the definition of a measure μ on $(X, \mathcal{M})$. Show that (i) can be replaced by the "nontriviality condition": There exists some $E \in \mathcal{M}$ with $\mu(E) < \infty$.

(6) Let $\{A_k\}$ be measurable and $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and

$$A = \{x \in X : x \in A_k \text{ for infinitely many } k\}.$$

(a) Show that A is measurable.

(b) Show that $\mu(A) = 0$.

This is Borel-Cantelli lemma, google for more.

(7) Let T be a map from a measure space $(X, \mathcal{M}, \mu)$ onto a set Y. Let $\mathcal{N}$ be the set of all subsets N of Y satisfying $T^{-1}(N) \in \mathcal{M}$. Show that the triple $(Y, \mathcal{N}, \lambda)$ where $\lambda(N) = \mu(T^{-1}(N))$ is a measure space.

(8) In Theorem 1.6 we approximate a non-negative measurable function f by an increasing sequence of simple functions from below. Can we approximate f by a decreasing sequence of simple functions from above? A necessary condition is that f must be bounded in X, that is, $f(x) \leq M$, $\forall x \in X$ for some M. Under this condition, show that this is possible.

(9) A measure space is *complete* if every subset of a *null set*, that is, a measurable set whose measure is equal to zero, is measurable. This problem shows that every measure space can be extended to become a complete measure at no additional price. It will be used later.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\widetilde{\mathcal{M}}$ contain all sets E such that there exist $A, B \in \mathcal{M}$, $A \subset E \subset B$, $\mu(B \setminus A) = 0$. Show that $\widetilde{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$ and if we set $\widetilde{\mu}(E) = \mu(A)$, then $(X, \widetilde{\mathcal{M}}, \widetilde{\mu})$ is a complete measure space.

(10) Here we review Riemann integral. This is an optional exercise. Let f be a bounded function defined on $[a, b], a, b \in \mathbb{R}$. Given any partition $P = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ on $[a, b]$ and tags $z_j \in [x_j, x_{j+1}]$, there corresponds a *Riemann sum* of f given by $R(f, P, \mathbf{z}) = \sum_{j=0}^{n-1} f(z_j)(x_{j+1} - x_j)$. The function f is called *Riemann integrable* with integral L if for every $\varepsilon > 0$ there exists some δ such that

$$\big|R(f, P, \mathbf{z}) - L\big| < \varepsilon,$$

whenever $\|P\| < \delta$ and $\mathbf{z}$ is any tag on P. (Here $\|P\| = \max_{j=0}^{n-1} |x_{j+1} - x_j|$ is the length of the partition.) Show that

(a) For any partition P, define its *Darboux upper* and *lower sums* by

$$\overline{R}(f,P) = \sum_j \sup\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j),$$

and

$$\underline{R}(f,P) = \sum_j \inf\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j)$$

respectively. Show that for any sequence of partitions $\{P_n\}$ satisfying $\|P_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \overline{R}(f,P_n)$ and $\lim_{n\to\infty} \underline{R}(f,P_n)$ exist.

(b) $\{P_n\}$ as above. Show that f is Riemann integrable if and only if

$$\lim_{n\to\infty} \overline{R}(f,P_n) = \lim_{n\to\infty} \underline{R}(f,P_n) = L.$$

(c) A set E in $[a,b]$ is called *of measure zero* if for every $\varepsilon > 0$, there exists a countable subintervals J_n satisfying $\sum_n |J_n| < \varepsilon$ such that $E \subset \bigcup_n J_n$. Prove Lebsegue's theorem which asserts that f is Riemann integrable if and only if the set consisting of all discontinuity points of f is a set of measure zero. Google for help if necessary.